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# ON DISCRETE AND DISCRETIZED NON-LINEAR ELASTIC STRUCTURES IN UNILATERAL CONTACT (STABILITY, UNIQUENESS AND VARIATIONAL PRINCIPLES)

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**Abstract**—In the present paper problems related to discrete and discretized non-linear elastic structures in unilateral contact with a rigid support are considered in the range of large displacements. A finite dimensional vector matrix description, based on the concepts of generalized stresses and strains, is derived. It is shown that the problem of determining the displacements and the contact forces for a given constant external loading can be formulated alternatively as a variational inequality, representing the principle of virtual work, or as a set of Kuhn–Tucker relations, representing force equilibrium. Furthermore, the Kuhn–Tucker relations are related to a primal and a dual minimization problem. The primal problem represents the principle of minimum of potential energy and the dual problem is a generalization to large displacements of the so-called reciprocal formulation of contact problems. Moreover, the problems of mechanical stability and that of uniqueness of incremental response are investigated. The incremental, or rate, formulation is derived together with an associated variational inequality, representing the incremental principle of virtual work. A sufficient condition for the uniqueness of the solution of this variational inequality is given. A sufficient condition for mechanical stability, on the other hand, can be obtained directly from a second-order sufficient condition for the optimum of non-linear programs. The fact that these two sufficient conditions do not coincide is discussed and a simple naturally discrete problem exemplifies this point. Furthermore, it is seen that the curvature of the rigid support has an influence on both the stability and the uniqueness of the structure. This fact is also illustrated by an example.

## 1. INTRODUCTION

Mechanical structures are frequently subject to constraints on their deformations. In nature these constraints are usually one-sided or unilateral. Nevertheless, mathematical models of mechanical systems frequently allow only for two-sided or bilateral constraints. The reason for this is perhaps that the mathematical theory needed to properly treat unilateral constraints is a fairly recent development. Today, however, tools capable of modelling unilateral constraints are available in two mathematical disciplines, one concentrates on finite dimensional problems and the other on infinite dimensional ones. For finite dimensional problems, inequality conditions, which are the proper mathematical description of unilateral mechanical constraints, have been thoroughly studied in the discipline of mathematical programming (MP). In solid and structural mechanics, unilateral problems of elastic–plastic material behaviour have been treated by these methods, both in the case of non-linear and linearized kinematical descriptions[1–3]. For the treatment of infinite dimensional problems involving inequality conditions, the theory of variational inequalities has been developed. This theory seems to have originated from studies of the mechanical problem of Signorini, which arises when a linear elastic body is in frictionless unilateral contact with a rigid support. Therefore, it is natural that in mechanics mainly various types of contact problems have been treated by these methods[4–6]. However, except for the recent paper by Ciarlet and Necas[7], the investigations have been limited to linearized kinematics.

The present investigation deals with a non-linear Signorini problem, i.e. a non-linear elastic solid in the range of large displacements (non-linear kinematics) which may come into frictionless unilateral contact with a rigid support. Based on the concepts of generalized stresses and strains[8–10], a finite dimensional description of the problem is derived. Thus, the resulting mathematical problem can be treated by the methods of MP. However, the usual emphasis in the MP literature is on the study of optimization problems set down *ab initio*: necessary and sufficient optimum conditions are derived in the analysis. In mechanics, on the other hand, the first-order necessary conditions (usually equilibrium conditions) are given at the outset and from these one derives various variational and optimization problems, usually known as mechanical "principles". This tradition is followed also in this paper and in that respect it is in agreement with the extensive work of Noble and Sewell[11]. However, the framework set up by them requires convexity of the global strain energy function, a property known not to hold in the present problem. Therefore, for the derivation of mechanical principles it was found convenient to rely on arguments based on variational inequalities, while the notation is that of vectors and matrices usually found in the MP literature. The connection with non-linear programming is confirmed when the equivalence between the principle of virtual work and a set of Kuhn–Tucker relations, the latter of which can be interpreted as equilibrium conditions, is shown.

Thus, the first objective of this paper is to derive a general discrete model for the non-linear Signorini problem, and the formulation of related mechanical principles such as those of virtual work and minimum of potential energy.

A second objective of the paper is the derivation of sufficient conditions for mechanical stability and uniqueness of incremental response, in the case of unilateral contact conditions. These are subjects that have been treated using MP methodology in the case of elastic–plastic material behaviour[12]. For the present problem it is shown that if a stable configuration is defined as one in which the potential energy attains a strict local minimum, then a second-order sufficient condition of non-linear programming can also be interpreted as a sufficient condition for mechanical stability. Regarding uniqueness of incremental response, the incremental, or rate, problem, which is a problem in terms of time derivatives of the displacement and force vectors, is formulated. A variational inequality formulation of this problem, representing the incremental principle of virtual work, is given. A sufficient condition for this problem to have a unique solution is easily obtained. Now, an interesting conclusion is that the two sufficient conditions for, respectively, stability and uniqueness, do not coincide. This is due to the presence of unilateral constraints. The situation is reminiscent of the classical one of "stable bifurcation", encountered in the theory of elastic-plastic bodies[13]. Furthermore, both stability and uniqueness is determined by a quadratic form, containing a square matrix. This matrix may be interpreted as what is known in finite element analysis as the tangential stiffness matrix. However, in the case of unilateral constraints it is found to contain a new sub-term, which is due to the contact force and the curvature of the rigid support. Thus, this curvature influences the stability of the structure, as shown in Fig. 2. Both the influence of the curvature of the support and the difference between the two sufficient conditions are exemplified by the study of explicit structures in the last section of this paper.

Finally, although the explicit statements of the results of this paper are restricted to the discrete formulation, the general features of mechanical behaviour disclosed should be expected to be found also for the underlying continuous problem. Moreover, if vector products shown by superscript T are interpreted as bilinear forms on vector fields and matrices replaced by differential operators (see, for instance Ref. [11] or Ref. [14]) the variational principles derived are valid also for continuous problems.

## 2. THE MATHEMATICAL MODEL

### 2.1. *Equations of discrete non-linear elasticity*

A most effective and elegant representation of the relations governing the mechanical behaviour of discrete and discretized elastic structures undergoing large displacements is based on the concepts of generalized stresses and strains. This representation has been extensively explored and refined in numerous papers by Argyris and co-workers under the name of the "natural approach". (See Ref. [8] and the references cited there.) It has also been presented and interpreted by Besseling[9, 10], using linear algebra as an ideal tool. Furthermore, Corradi[15] used generalized variables to provide a rational basis for understanding the problem of stress computation in displacement finite element models. In this section a short discussion is given on how to deduce a discrete representation in terms of generalized variables from a material, or Lagrangian, continuous field description of non-linear elasticity. The existence of a natural, or stress free, reference configuration is assumed in which the body occupies a region $B_0$, with boundary $\partial B_0$, in physical space.

Consider a finite element $e$, which in the reference configuration has volume $B_0^e \subset B_0$ and boundary $\partial B_0^e$. Mechanical equilibrium for the element can be expressed by the following virtual work equation:

$$\int_{\partial B_0^e} t_{0i}\delta u_i \, \mathrm{d}A_0 + \int_{B_0^e} \rho_0 f_i \delta u_i \, \mathrm{d}V_0 = \int_{B_0^e} S_{ij}\delta E_{ij} \, \mathrm{d}V_0. \tag{1}$$

Here, index notation referring to a system of Cartesian material coordinates $\mathbf{a} = (a_1, a_2, a_3)$ is adopted. The symbols used in eqn (1) are $t_{0i}$ the surface tractions acting in the current configuration but measured per unit area of the reference configuration; $\rho_0$ the mass density of the reference configuration; $f_i$ the body forces per unit mass; $S_{ij}$ the components of the second Piola–Kirchhoff stress tensor; $u_i$ the displacements; $E_{ij}$ the components of the Green strain tensor; $\delta$ the variational operator. The components of the Green strain tensor are related to the displacement components by the equation

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i} + \frac{\partial u_k}{\partial a_i}\frac{\partial u_k}{\partial a_j}\right). \tag{2}$$

Through the finite element interpolation functions $\psi_N^e(\mathbf{a})$, the displacement field $u_i(\mathbf{a})$, within the finite element, can be expressed in terms of the nodal displacements $u_{iN}^e$

$$u_i(\mathbf{a}) = \sum_{N=1}^{n} \psi_N^e(\mathbf{a}) u_{iN}^e. \tag{3}$$

By introducing eqn (3) on the left-hand side of eqn (1) one obtains

$$\text{L.H.S. } (1) = \mathbf{F}^{e\,T}\delta\mathbf{u}^e \tag{4}$$

where $\mathbf{u}$ is a vector of nodal displacements, $\mathbf{F}^e$ is a vector of consistent nodal forces and superscript T denotes the transpose of a vector. To be able to treat the right-hand side of eqn (1) similar to the left-hand side, the concept of generalized strains needs to be introduced. This concept hinges on the observation that if the *displacement* of a finite element can be described by a finite number of parameters (the nodal displacements), then it certainly must be possible to also describe *deformation* in the same way. The parameters that describe the deformation are the generalized strains, and the number of such strains equals the number of nodal displacements minus the number of rigid body freedoms. The generalized strains can be collected into a vector $\boldsymbol{\varepsilon}^e$, which is related to the nodal displacements through a non-linear algebraic equation

$$\boldsymbol{\varepsilon}^e = \mathbf{B}^e(\mathbf{u}^e) \tag{5}$$

which is such that $\boldsymbol{\varepsilon}$ vanishes if and only if the finite element performs rigid body displacements. Equation (5) replaces eqn (2) in the case of a discrete representation.

For the actual realization of eqn (5), Argyris *et al.*[8] identifies the generalized strains with certain geometrically well-defined deformation measures of the finite element. For instance, in the case of a triangular element the elongations of its edges may serve as generalized strains. Besseling[9], on the other hand, suggests that eqn (5) can be obtained by introducing eqn (3) into eqn (2) and applying the resulting equation at a discrete number of points within the finite element. This procedure gives a close connection between numerical integration and generalized strains[15]. Furthermore, Besseling's approach suggests that interpolation functions $\phi_{Kij}^e(\mathbf{a})$ can be introduced, such that the strain field $E_{ij}(\mathbf{a})$, within the finite element can be expressed in terms of the components $\varepsilon_K$ of $\boldsymbol{\varepsilon}^e$

$$E_{ij}(\mathbf{a}) = \sum_{K=1}^{k} \phi^e_{Kij}(\mathbf{a})\varepsilon^e_K. \tag{6}$$

Introducing eqn (6) into the right-hand side of eqn (1) gives

$$\text{R.H.S. (1)} = \boldsymbol{\sigma}^{e\mathrm{T}}\delta\boldsymbol{\varepsilon}^e \tag{7}$$

where the components $\sigma^e_K$ of $\boldsymbol{\sigma}^e$ are the generalized stresses, defined by

$$\sigma^e_K = \int_{B^e_0} S_{ij}\phi^e_{Kij}(\mathbf{a}) \, \mathrm{d}V_0. \tag{8}$$

The virtual work eqn (1) can now be written as

$$\mathbf{F}^{e\mathrm{T}}\delta\mathbf{u}^e = \boldsymbol{\sigma}^{e\mathrm{T}}\delta\boldsymbol{\varepsilon}^e, \quad \forall\,(\delta u^e, \delta\boldsymbol{\varepsilon}^e) \in \delta V^e(\mathbf{u}^e) \tag{9}$$

where

$$\delta V^e(\mathbf{u}^e) = \{(\delta\mathbf{u}^e, \delta\boldsymbol{\varepsilon}^e) \mid \delta\boldsymbol{\varepsilon}^e = \nabla\mathbf{B}^e(\mathbf{u}^e)\delta\mathbf{u}^e\} \tag{10}$$

is the set of kinematically admissible variations. The notation $\nabla\mathbf{B}^e(\mathbf{u}^e)$ denotes the Jacobian of the non-linear eqn (5). Since the components of $\delta\mathbf{u}^e$ are independent, eqn (9) implies

$$\mathbf{F}^e = \nabla\mathbf{B}^e(\mathbf{u}^e)^{\mathrm{T}}\boldsymbol{\sigma}^e. \tag{11}$$

Equations (5) and (11) represent the kinematic and the static conditions of the finite element, respectively. However, to fully characterize the mechanical behaviour a constitutive equation, connecting generalized stresses with generalized strains is also needed. To that end one will be restricted to hyperelastic behaviour, implying the existence of a strain energy function, $\mathscr{W} = \mathscr{W}(\mathbf{E})$, per unit mass of the reference configuration, such that

$$S_{ij} = \rho_0 \frac{\partial \mathscr{W}(\mathbf{E})}{\partial E_{ij}}. \tag{12}$$

By using the strain interpolation (6) one can determine a strain energy for the finite element as a whole

$$W^e = \int_{B^e_0} \rho_0 \mathscr{W}(\mathbf{E}) \, \mathrm{d}V_0. \tag{13}$$

It follows that

$$\frac{\partial W^e}{\partial \varepsilon^e_K} = \int_{B^e_0} \rho_0 \frac{\partial \mathscr{W}}{\partial E_{ij}} \phi^e_{Kij} \, \mathrm{d}V_0. \tag{14}$$

From eqn (12) and by comparing with eqn (8) one obtains

$$\boldsymbol{\sigma}^e = \nabla W^e(\boldsymbol{\varepsilon}^e). \tag{15}$$

That is, the vector of generalized stresses is obtained as the gradient of the strain energy of the element, when expressed in terms of the generalized strains.

The mechanical behaviour of a finite element is now fully described by eqns (5), (11) and (15). The next step in building a discrete theory for non-linear elasticity consists in assembling the finite elements to form a mechanical structure. The vectors of generalized variables for this structure are formed as the direct sums of the individual vectors, i.e. in the case of $n$ finite elements

$$\boldsymbol{\sigma}^{\mathrm{T}} = [\boldsymbol{\sigma}^{1\mathrm{T}}, \boldsymbol{\sigma}^{2\mathrm{T}}, \ldots, \boldsymbol{\sigma}^{n\mathrm{T}}]$$
$$\boldsymbol{\varepsilon}^{\mathrm{T}} = [\boldsymbol{\varepsilon}^{1\mathrm{T}}, \boldsymbol{\varepsilon}^{2\mathrm{T}}, \ldots, \boldsymbol{\varepsilon}^{n\mathrm{T}}]. \quad (16)$$

The total strain energy is the sum

$$W = \sum_{e=1}^{n} W^{e}. \quad (17)$$

Equation (15) then implies

$$\boldsymbol{\sigma} = \nabla W(\boldsymbol{\varepsilon}). \quad (18)$$

The configuration of the structure is described by a vector of nodal displacements $\mathbf{u}$. How the finite elements are assembled to form the mechanical structure is defined by $n$ matrices $\mathbf{T}^{e}$ such that

$$\mathbf{u}^{e} = \mathbf{T}^{e}\mathbf{u}. \quad (19)$$

Introducing these equations into eqn (5) and taking into account eqns (16) one obtains

$$\boldsymbol{\varepsilon} = \mathbf{B}(\mathbf{u}) \quad (20)$$

where it is understood that

$$\mathbf{B}(\mathbf{u})^{\mathrm{T}} = [\mathbf{B}^{1}(\mathbf{T}^{1}\mathbf{u})^{\mathrm{T}}, \ldots, \mathbf{B}^{n}(\mathbf{T}^{n}\mathbf{u})^{\mathrm{T}}].$$

Finally, motivated by the invariance of virtual work, a structural nodal force vector is defined as the sum

$$\mathbf{F} = \sum_{e=1}^{n} \mathbf{T}^{e\mathrm{T}}\mathbf{F}^{e}. \quad (21)$$

Equation (11) can then be extended to the whole structure as

$$\mathbf{F} = \nabla \mathbf{B}(\mathbf{u})^{\mathrm{T}}\boldsymbol{\sigma}. \quad (22)$$

The set of eqns (18), (20) and (22) now represents a matrix formulation of the mechanical behaviour of a non-linear elastic structure undergoing large displacements.

### 2.2. *Relations of unilateral contact*

The deformation of the body (structure) in the previous section will be considered to be constrained by the presence of a rigid surface $G(\mathbf{x}) = 0$ in physical space. $G(\mathbf{x})$ is assumed to be a smooth function defined over all of the space. The deformation $\mathbf{u}(\mathbf{a})$ of all material points $\mathbf{a}$ should be such that

$$G(\mathbf{a}+\mathbf{u}(\mathbf{a})) \leqslant 0. \quad (23)$$

When the body makes contact with the rigid surfaces, i.e. equality holds for some $\mathbf{a}$ in relation (23), a contact traction vector $\mathbf{q}$ arises. If the contact is frictionless, there is no traction in the tangential direction of the deformed body (which is assumed well defined) and one can write

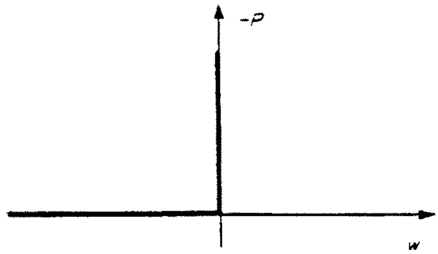


Fig. 1. Representation of unilateral conditions (23) and (25).

$$\mathbf{q}(\mathbf{a}) = \nabla G(\mathbf{a}+\mathbf{u}(\mathbf{a}))P(\mathbf{a}). \tag{24}$$

Here $P(\mathbf{a})$ is a scalar field, which can be considered defined on that part of the reference surface $\partial B_0$ that may potentially come into contact with the rigid surface. This part of the surface is denoted by $C_0$. If $G(\mathbf{x})$ is normalized so that $|\nabla G(\mathbf{x})| = 1$, for points $\mathbf{x}$ such that $G(\mathbf{x}) = 0$, $P(\mathbf{a})$ can be regarded as a field of contact pressure. From physical considerations the following constitutive conditions for $P(\mathbf{a})$ at all points $\mathbf{a}$ of $C_0$ can be suggested

$$P(\mathbf{a}) \leqslant 0, \quad P(\mathbf{a})G(\mathbf{a}+\mathbf{u}(\mathbf{a})) = 0. \tag{25}$$

That is, the contact pressure is compressive and different from zero only if the body is in contact with the rigid surface. It is useful to define a contact displacement field on $C_0$ as

$$w(\mathbf{a}) = G(\mathbf{a}+\mathbf{u}(\mathbf{a})). \tag{26}$$

Relations (23) and (25) then represent the unilateral type of behaviour in Fig. 1, which in the linearized theory of frictionless contact is assumed for the "normal" components of $\mathbf{q}$ and $\mathbf{u}$[5]. The present discussion shows that $P(\mathbf{a})$ and $w(\mathbf{a})$ are to be regarded as "normal" components in the case of large displacements.

Furthermore, the introduction of contact displacement (26) means that eqn (24) can be described by a virtual work type relation

$$\int_{C_0} P\delta w \, \mathrm{d}A_0 = \int_{C_0} q_i \delta u_i \, \mathrm{d}A_0 \tag{27}$$

where $w$ and $u_i$ are related by eqn (26). Similar to the way the virtual work eqn (1) in the previous section was used, eqn (27) will provide a basis for a discrete approximation of the equations representing the mechanical contact.

It is assumed that a finite element discretization according to the previous section has been performed. If the conditions representing mechanical contact are to be consistent with this discretization they should be expressed in a finite number of variables. Therefore, one only requires the satisfaction of relation (23) at a finite number of points of $C_0$. A natural choice of such points is those nodal points of the finite element discretization that are located on $C_0$. Although this is not the only choice possible, it seems as if other choices are likely to produce unstable finite element approximations[16]. Let the subvector of $\mathbf{u}$, associated with nodes located on $C_0$, be denoted by $\mathbf{u}_c$. Application of eqn (26) at the nodal points of $C_0$ results in the following non-linear algebraic equation:

$$\mathbf{w} = \mathbf{G}(\mathbf{u}_c). \tag{28}$$

A similar equation results if one applies eqn (26) at other points of $C_0$.

Contact displacements between nodal points can be approximated by applying the same interpolation for $w(\mathbf{a})$ as for displacements within the finite elements. That is, if the interpolation functions are extended as zero outside their respective elements, it holds for the whole of $C_0$ that

$$w(\mathbf{a}) = \sum_{e=1}^{l} \sum_{N=1}^{m_e} \psi_N^e(\mathbf{a}) w_N^e \tag{29}$$

where l is the number of finite elements with part of their boundary located on $C_0$ and $m_e$ is the number of nodal points on the contact boundary of such an element. $w_N^e$ are the components of $\mathbf{w}$.

Equation (29) can be used to rewrite the left-hand side of eqn (27)

$$\text{L.H.S. (27)} = \mathbf{P}^{\mathrm{T}} \delta \mathbf{w} \tag{30}$$

where the components $P_N^e$ of the vector $\mathbf{P}$ are defined by

$$P_N^e = \int_{C_0} P \psi_N^e(\mathbf{a}) \, \mathrm{d}A_0. \tag{31}$$

Comparing this equation with eqn (8) one can see that the proper name for $\mathbf{P}$ would be the vector of generalized contact pressure. However, the attribute generalized in the sequel will not be used.

The right-hand side of eqn (27) can be rewritten by introducing eqn (3) for each finite element with a boundary located on $C_0$. One obtains

$$\text{R.H.S. (27)} = \mathbf{F}_c^{\mathrm{T}} \delta \mathbf{u}_c. \tag{32}$$

where $\mathbf{F}_c$ is a vector of contact forces, calculated from $\mathbf{q}$ in a way consistent with the finite element displacement approximation. Obviously, $\mathbf{F}_c$ can be considered as a subvector of the vector $\mathbf{F}$ introduced in the previous section.

By using eqns (30) and (32), eqn (27) can be written as

$$\mathbf{P}^{\mathrm{T}} \delta \mathbf{w} = \mathbf{F}_c^{\mathrm{T}} \delta \mathbf{u}_c, \quad \forall\, (\delta \mathbf{w}, \delta \mathbf{u}_c) \in \delta V_c(\mathbf{u}_c) \tag{33}$$

where

$$\delta V_c(\mathbf{u}_c) = \{(\delta \mathbf{w}, \delta \mathbf{u}_c) \mid \delta \mathbf{w} = \nabla \mathbf{G}(\mathbf{u}_c) \delta \mathbf{u}_c\}.$$

Variational statement (33) implies that

$$\mathbf{F}_c = \nabla \mathbf{G}(\mathbf{u}_c)^{\mathrm{T}} \mathbf{P}. \tag{34}$$

This represents eqn (24) in the discrete theory.

By now it should be apparent that, in analogy with the previous section, eqns (28) and (34) play the role of kinematic and static conditions for the contact surface, respectively. Components of $\mathbf{P}$ and $\mathbf{w}$ are generalized variables in the discrete theory and should be related by constitutive relations. As constitutive relations for the underlying field variables $P$ and $w$ one can consider the complementarity conditions

$$w(\mathbf{a}) \leqslant 0, \quad P(\mathbf{a}) \leqslant 0, \quad P(\mathbf{a}) w(\mathbf{a}) = 0 \tag{35}$$

which are valid for all points $\mathbf{a}$ of $C_0$. As already discussed, relation $(35)_1$ is in the discrete theory replaced by $\mathbf{w} \leqslant \mathbf{0}$. Conditions $(35)_2$ and $(35)_3$ are, perhaps, not as obviously replaced. However, what should be noticed is that $\mathbf{P}$ is defined through a virtual work equivalence (30). Therefore, relations (35) are rewritten as a variational inequality in terms of the virtual work. It can be shown[5] that relations (35) are equivalent, at least for sufficiently smooth fields, to the requirement that

$$\int_{C_0} P\delta w \, dA_0 \geq 0 \tag{36}$$

for all fields $\delta w(\mathbf{a})$ such that $w(\mathbf{a})+\delta w(\mathbf{a}) \leq 0$ everywhere on $C_0$. The virtual work equivalence (30) then implies that relations (35) become

$$\mathbf{w} \leq \mathbf{0}, \quad \mathbf{P} \leq \mathbf{0}, \quad \mathbf{P}^T\mathbf{w} = 0. \tag{37}$$

The unilateral contact conditions for a discrete structure undergoing large displacements is now described by the set of relations (28), (34) and (37).

## 3. VARIATIONAL INEQUALITIES

In the previous section a mathematical model that describes the mechanical behaviour of unilaterally supported elastic structures has been formulated. For a given external loading on the structure this model forms the mathematical problem of determining the displacements and the contact pressure. It will be shown in this section that this problem can be stated as a variational inequality, which has the mechanical meaning of the principle of virtual work. Furthermore, by using the duality theory of linear programming, this variational inequality can be shown equivalent to a set of relations, which in the next section will be identified with the Kuhn–Tucker conditions of a certain minimization problem and which have the mechanical meaning of force equilibrium. The Lagrangian multipliers of these conditions can be identified with the contact forces.

The fact that $\mathbf{F}_c$ and $\mathbf{u}_c$ are subvectors of $\mathbf{F}$ and $\mathbf{u}$, respectively, is first stressed. Therefore, by denoting the complement of $\mathbf{F}_c$ by $\mathbf{F}_2$, eqn (22) can be written as

$$\mathbf{F}_c = \nabla\mathbf{B}^1(\mathbf{u})^T\boldsymbol{\sigma} \tag{38}$$

$$\mathbf{F}_2 = \nabla\mathbf{B}^2(\mathbf{u})^T\boldsymbol{\sigma} \tag{39}$$

where an obvious decomposition of $\nabla\mathbf{B}(\mathbf{u})$ has been used. Similarly, by denoting the complement of $\mathbf{u}_c$ by $\mathbf{u}_2$, the variational form of eqn (20) becomes

$$\delta\boldsymbol{\varepsilon} = \nabla\mathbf{B}^1(\mathbf{u})\delta\mathbf{u}_c + \nabla\mathbf{B}^2(\mathbf{u})\delta\mathbf{u}_2. \tag{40}$$

The principle of virtual work for the free structure (i.e. without considering the unilateral support), which is a statement equivalent to eqns (38) and (39), can now be written as

$$\boldsymbol{\sigma}^T\delta\boldsymbol{\varepsilon} = \mathbf{F}_c^T\delta\mathbf{u}_c + \mathbf{F}_2^T\delta\mathbf{u}_2, \quad \forall\, (\delta\boldsymbol{\varepsilon}, \delta\mathbf{u}) \in \delta V(\mathbf{u}) \tag{41}$$

where

$$\delta V(\mathbf{u}) = \{(\delta\boldsymbol{\varepsilon}, \delta\mathbf{u}) \mid (40)\}. \tag{42}$$

Furthermore, the complementarity conditions (37) are equivalent to the variational inequality

$$\mathbf{w} \leq \mathbf{0}, \quad \mathbf{P}^T\delta\mathbf{w} \geq 0, \quad \forall\, \delta\mathbf{w} \in \delta K(\mathbf{w}) \tag{43}$$

where

$$\delta K(\mathbf{w}) = \{\delta\mathbf{w} \mid \mathbf{w}+\delta\mathbf{w} \leqslant \mathbf{0}\}. \tag{44}$$

From relations (28), (33), (41) and (43) the following material-independent statement of the principle of virtual work is obtained:

$$\mathbf{G}(\mathbf{u}_c) \leqslant \mathbf{0}, \quad \boldsymbol{\sigma}^T\delta\boldsymbol{\varepsilon} \geqslant \mathbf{F}_2^T\delta\mathbf{u}_2, \quad \forall\,(\delta\boldsymbol{\varepsilon},\delta\mathbf{u}) \in \delta K^1(\mathbf{u}) \tag{45}$$

where

$$\delta K^1(\mathbf{u}) = \{(\delta\boldsymbol{\varepsilon},\delta\mathbf{u}) \mid (40),\quad \mathbf{G}(\mathbf{u}_c)+\nabla\mathbf{G}(\mathbf{u}_c)\delta\mathbf{u}_c \leqslant \mathbf{0}\}. \tag{46}$$

Assume now that the internal forces are conservative in the way described by eqn (18) and that $\mathbf{F}_2$ represents external constant, or "death", loads. Then the variational inequality (45) results in the following problem.

*Problem* 1. Find $\mathbf{u} \in \mathscr{K}^1$ such that

$$\delta\Pi[\mathbf{u}] \geqslant 0, \quad \forall\,\delta\mathbf{u} \in \delta\mathscr{K}^1(\mathbf{u}) \tag{47}$$

where

$$\Pi[\mathbf{u}] = W(\mathbf{B}(\mathbf{u})) - \mathbf{F}_2^T\mathbf{u}_2 \tag{48}$$

is the potential energy of the structure. The set

$$\mathscr{K}^1 = \{\mathbf{u} \mid \mathbf{G}(\mathbf{u}_c) \leqslant \mathbf{0}\} \tag{49}$$

represents the kinematically admissible displacements and the set

$$\delta\mathscr{K}^1(\mathbf{u}) = \{\delta\mathbf{u} \mid \mathbf{G}(\mathbf{u}_c)+\nabla\mathbf{G}(\mathbf{u}_c)\delta\mathbf{u}_c \leqslant \mathbf{0}\} \tag{50}$$

which may be considered derivable from $\mathscr{K}^1$ using the rules of variational calculus, represents the kinematically admissible variations of displacements.

Problem 1 is the material-dependent principle of virtual work which characterizes an equilibrium configuration of the structure. Note that due to the unilateral constraints the potential energy is not stationary in this equilibrium configuration. Rather, it satisfies the conditions of a substationary point[6].

An equivalent of Problem 1 is to find $\mathbf{u} \in \mathscr{K}^1$ such that the linear programming problem ($\mathbf{u}$ is fixed and $\delta\mathbf{u}$ is the variable)

$$\begin{aligned} &\text{minimize} \quad \delta\Pi[\mathbf{u}] \\ &\text{subject to} \quad \delta\mathbf{u} \in \delta\mathscr{K}^1(\mathbf{u}) \end{aligned} \tag{51}$$

has the solution zero. According to the duality theorem of linear programming[17] this problem has the dual

$$\begin{aligned} &\text{maximize} \quad \boldsymbol{\lambda}^T\mathbf{G}(\mathbf{u}_c) \\ &\text{subject to} \quad \mathbf{B}_u(u)^T\nabla W(\mathbf{B}(\mathbf{u})) + \begin{bmatrix} \nabla\mathbf{G}(\mathbf{u}_c)^T\boldsymbol{\lambda} \\ -\mathbf{F}_2 \end{bmatrix} = \mathbf{0}, \quad \boldsymbol{\lambda} \geqslant \mathbf{0} \end{aligned}$$

the solution of which is also zero. Thus, for a $\mathbf{u}$ that solves Problem 1 there exists a vector $\boldsymbol{\lambda}$ such that the following relations are satisfied:

$$\mathbf{B}_u(\mathbf{u})^T\nabla W(\mathbf{B}(\mathbf{u}))+\begin{bmatrix}\nabla \mathbf{G}(\mathbf{u}_c)^T\boldsymbol{\lambda}\\ -\mathbf{F}_2\end{bmatrix}=\mathbf{0} \tag{52}$$

$$\boldsymbol{\lambda}\geqslant\mathbf{0},\quad \mathbf{G}(\mathbf{u}_c)\leqslant\mathbf{0},\quad \boldsymbol{\lambda}^T\mathbf{G}(\mathbf{u}_c)=0. \tag{53}$$

If the identification $\boldsymbol{\lambda} = -\mathbf{P}$ is made, relations (52) and (53) can be derived from the relations of Section 2 by direct substitution. Thus, one can see that the solutions of Problem 1 are the same as those satisfying the discrete equations of Section 2.

Finally, one can remark that solutions $\mathbf{u}$ and $\mathbf{P} = -\boldsymbol{\lambda}$ of relations (52) and (53) can be characterized by the following mixed type of variational inequality problem.

*Problem* 2. Find $(u, P)\in\mathscr{K}^2$ such that

$$\delta L[\mathbf{u},\mathbf{P}]\leqslant 0,\quad \forall\,(\mathbf{u},\mathbf{P})\in\delta\mathscr{K}^2(\mathbf{P}) \tag{54}$$

where

$$L[\mathbf{u},\mathbf{P}] = W(\mathbf{B}(\mathbf{u}))-\mathbf{P}^T\mathbf{G}(\mathbf{u}_c)-\mathbf{F}_2^T\mathbf{u}_2 \tag{55}$$

$$\mathscr{K}^2 = \{(\mathbf{u},\mathbf{P})\,|\,\mathbf{P}\leqslant\mathbf{0}\} \tag{56}$$

$$\delta\mathscr{K}^2(\mathbf{P}) = \{(\delta\mathbf{u},\delta\mathbf{P})\,|\,\mathbf{P}+\delta\mathbf{P}\leqslant\mathbf{0}\}. \tag{57}$$

The function $L[\mathbf{u},\mathbf{P}]$ is the Lagrangian function that will be of central importance in the sequel.

## 4. OPTIMIZATION PROBLEMS

From the theory of non-linear programming it is known that a necessary condition (under certain regularity assumptions[17]) for a point to be a solution of a constrained minimization problem is expressed by the Kuhn–Tucker conditions. Consider the problem of finding the minimum of the potential energy.

*Problem* 3. Find $u\in\mathscr{K}^1$ such that

$$\Pi[\mathbf{u}] = \min\,\{\Pi[\mathbf{u}^*]\,|\,\mathbf{u}^*\in\mathscr{K}^1\}. \tag{58}$$

One finds that the Kuhn–Tucker conditions of this problem are identical with relations (52) and (53), which were shown to characterize all equilibrium conditions of the discrete unilaterally supported structure. Furthermore, as will be discussed in Section 5, a strict local minimum of the potential energy corresponds to a mechanically stable equilibrium configuration. Hence, the interest for Problem 3 is obvious.

Here briefly the possibility of characterizing the Lagrangian multiplier vector $\boldsymbol{\lambda} = -\mathbf{P}$ as the solution of a dual optimization problem is discussed. (Problem 3 is considered as the primal problem.) Assume that $\mathbf{u}$ is a local solution of Problem 3. Then there exists a vector $\boldsymbol{\lambda}$ related to $\mathbf{u}$ through relations (52) and (53), and it is known from the second-order necessary conditions[17], that at such a solution point the Hessian of Lagrangian (55) is positive semidefinite ($\mathbf{P}$ considered fixed) on the tangent subspace of the active constraints. However, in order to make a dual function well defined the stronger assumption that the Hessian of the Lagrangian is positive definite on the whole Euclidean space is made. Locally, near $\mathbf{u}$, one can then define the dual function

$$R(\mathbf{P}^*) = \min_{\mathbf{u}^*} L[\mathbf{u}^*,\mathbf{P}^*]. \tag{59}$$

From the local duality theorem[17] Problem 4 is obtained.

*Problem* 4. Find $\mathbf{P} \leqslant \mathbf{0}$ such that

$$R(\mathbf{P}) = \min \{R(\mathbf{P}^*) \mid \mathbf{P}^* \leqslant \mathbf{0}\} \tag{60}$$

has a local solution if Problem 3 has a local solution where the Hessian of the Lagrangian is positive definite. The two solutions are related by the Kuhn–Tucker conditions (52) and (53).

The interest in Problem 4 is due to the fact that it is a generalization to the non-linear case of the so-called reciprocal formulation of frictionless contact problems[18, 19], that is, a formulation in terms of contact forces obtained by explicitly evaluating the minimum in eqn (59). Indeed, in the case when the strain energy is given by

$$W = \tfrac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{E}\boldsymbol{\varepsilon} \tag{61}$$

and

$$\mathbf{B}_u(\mathbf{u}) = \mathbf{B}(\mathbf{u}) = \mathbf{B}\mathbf{u} \tag{62}$$

$$\nabla\mathbf{G}(\mathbf{u}_c) = \mathbf{G}(\mathbf{u}_c) = \mathbf{G}\mathbf{u}_c \tag{63}$$

are linear, the dual function can be written as[19]

$$R(\mathbf{P}) = \tfrac{1}{2}\mathbf{P}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{C}\mathbf{P} + \mathbf{C}^{\mathrm{T}}\mathbf{K}^{-1}\begin{bmatrix}\mathbf{0}\\ \bar{\mathbf{F}}_2\end{bmatrix} \tag{64}$$

where

$$\mathbf{C} = \begin{bmatrix}\mathbf{G}^{\mathrm{T}}\\ \mathbf{0}\end{bmatrix}, \quad \mathbf{K} = \mathbf{B}^{\mathrm{T}}\mathbf{E}\mathbf{B}. \tag{65}$$

## 5. THE INCREMENTAL PROBLEM

This section treats the incremental or rate problem associated with the mechanical problem under consideration. In a quasi-static loading process the displacements and contact pressures are supposed to have been determined up to a time $t$. The incremental changes of these variables are to be calculated for a further infinitesimal variation of the external loads.

The equations governing this incremental problem are obtained by Taylor expansions of the basic equations of Section 2. Using notations

$$\mathbf{u}(t+\mathrm{d}t) = \mathbf{u}(t) + \mathrm{d}\mathbf{u}(t) = \bar{\mathbf{u}} + \dot{\mathbf{u}}\,\mathrm{d}t \tag{66}$$

and similarly for other vectors, eqns (18), (20) and (22) give

$$\bar{\boldsymbol{\sigma}} + \dot{\boldsymbol{\sigma}}\,\mathrm{d}t = \nabla W(\bar{\boldsymbol{\varepsilon}}) + \nabla^2 W(\bar{\boldsymbol{\varepsilon}})\dot{\boldsymbol{\varepsilon}}\,\mathrm{d}t \tag{67}$$

$$\bar{\boldsymbol{\varepsilon}} + \dot{\boldsymbol{\varepsilon}}\,\mathrm{d}t = \mathbf{B}(\bar{\mathbf{u}}) + \nabla\mathbf{B}(\bar{\mathbf{u}})\dot{\mathbf{u}}\,\mathrm{d}t \tag{68}$$

$$\bar{\mathbf{F}} + \dot{\mathbf{F}}\,\mathrm{d}t = \nabla\mathbf{B}(\bar{\mathbf{u}})^{\mathrm{T}}\bar{\boldsymbol{\sigma}} + \nabla\mathbf{B}(\bar{\mathbf{u}})^{\mathrm{T}}\dot{\boldsymbol{\sigma}}\,\mathrm{d}t + \bar{\boldsymbol{\sigma}}^{\mathrm{T}}\nabla^2\mathbf{B}(\bar{\mathbf{u}})\dot{\mathbf{u}}\,\mathrm{d}t \tag{69}$$

where the following notation is used :

$$\boldsymbol{\lambda}^T \nabla^2 \mathbf{J}(\mathbf{x}) = \sum_r \lambda_r \nabla^2 J_r(\mathbf{x}) \tag{70}$$

for a sum of Hessians. Similarly, from eqns (28) and (34) one obtains

$$\bar{\mathbf{w}} + \dot{\mathbf{w}}\,\mathrm{d}t = \mathbf{G}(\bar{\mathbf{u}}_c) + \nabla\mathbf{G}(\bar{\mathbf{u}}_c)\dot{\mathbf{u}}_c\,\mathrm{d}t \tag{71}$$

$$\bar{\mathbf{F}}_c + \dot{\mathbf{F}}_c\,\mathrm{d}t = \mathbf{G}(\bar{\mathbf{u}}_c)^T\bar{\mathbf{P}} + \nabla\mathbf{G}(\bar{\mathbf{u}}_c)^T\dot{\mathbf{P}}\,\mathrm{d}t + \bar{\mathbf{P}}^T\nabla^2\mathbf{G}(\bar{\mathbf{u}}_c)\dot{\mathbf{u}}_c\,\mathrm{d}t. \tag{72}$$

Since zero-order terms cancel equations for rates from eqns (67)–(69), (71) and (72) are obtained. By simple substitution $\dot{\boldsymbol{\varepsilon}}$, $\dot{\boldsymbol{\sigma}}$, $\dot{\mathbf{w}}$ and $\dot{\mathbf{F}}_c$ are eliminated from these equations and one obtains

$$\left\{\nabla\mathbf{B}(\bar{\mathbf{u}})^T\nabla^2 W(\mathbf{B}(\bar{\mathbf{u}}))\nabla\mathbf{B}(\bar{\mathbf{u}}) + \nabla W(\mathbf{B}(\bar{\mathbf{u}}))^T\nabla^2\mathbf{B}(\bar{\mathbf{u}}) - \left[\begin{array}{c|c} \bar{\mathbf{P}}^T\nabla^2\mathbf{G}(\bar{\mathbf{u}}_c) & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0}\end{array}\right]\right\}\dot{\mathbf{u}} = \left[\begin{array}{c} \nabla\mathbf{G}(\bar{\mathbf{u}}_c)^T\dot{\mathbf{P}} \\ \hline \dot{\mathbf{F}}_2 \end{array}\right]. \tag{73}$$

The matrix within the curly brackets { } can be interpreted as a tangential stiffness matrix. The second of its three terms is the well-known geometric stiffness matrix, which is due to initial stresses $\bar{\boldsymbol{\sigma}} = \nabla W(\mathbf{B}(\bar{\mathbf{u}}))$. The third term, on the other hand, seems to be encountered for the first time in this paper, although it is similar to the stiffness matrix discussed by Wriggers and Simo[20] in the context of two-body contact problems. It is similar to the geometric stiffness matrix but is due to initial contact forces $\bar{\mathbf{P}}$ and depends on the curvature of the rigid surface. Its importance for mechanical stability and uniqueness of incremental response will be discussed in the next section.

Complementarity conditions (37) introduce constraints on increments $\dot{w}$ and $\dot{P}$ at time $t$. To give these constraints $I$ is denoted as the set of indices of nodal points on the contact boundary. At time $t$, $I$ can be divided into three subsets in accordance with the present values of contact displacements and pressures

$$i \in A \subset I \text{ iff } w_i(t) < 0, \quad P_i(t) = 0$$

$$i \in B \subset I \text{ iff } w_i(t) = 0, \quad P_i(t) = 0$$

$$i \in C \subset I \text{ iff } w_i(t) = 0, \quad P_i(t) < 0.$$

The conditions imposed on the increments by the complementarity conditions can now be given by

(i) if $i \in A$

$$\dot{w}_i = \nabla G_i(\bar{\mathbf{u}}_c)\dot{\mathbf{u}}_c \lesseqgtr 0, \quad \dot{P}_i = 0\,; \tag{74}$$

(ii) if $i \in B$

$$\dot{w}_i = \nabla G_i(\bar{\mathbf{u}}_c)\dot{\mathbf{u}}_c \leqslant 0, \quad \dot{P}_i \leqslant 0, \quad \dot{P}_i\nabla G_i(\mathbf{u}_c)\dot{\mathbf{u}}_c = \dot{P}_i\dot{w}_i = 0\,; \tag{75}$$

(iii) if $i \in C$

$$\dot{w}_i = \nabla G_i(\bar{\mathbf{u}}_c)\dot{\mathbf{u}}_c = 0, \quad \dot{P}_i \lesseqgtr 0. \tag{76}$$

Relations (73)–(76) can be interpreted as Kuhn–Tucker conditions of similar importance for the incremental problem as relations (52) and (53) for the finite displacement problem in Section 3. Obviously, several variational inequality problems can be shown to be equivalent to relations (73)–(76) and if the tangential stiffness matrix $\mathbf{K} = \{\ \}$ is positive definite a set of primal–dual extremum principles exists. However, these subjects will not be elaborated on here except one variational inequality formulation, equivalent to relations (73)–(76), which will be of importance in the next section.

*Problem* 5. Find $\dot{u} \in \dot{\mathscr{K}}(\bar{u})$ such that

$$\delta\dot{\Pi}[\bar{\mathbf{u}}, \dot{\mathbf{u}}] \geqslant 0, \quad \forall\, \delta\dot{\mathbf{u}} \in \delta\dot{\mathscr{K}}(\bar{\mathbf{u}}, \dot{\mathbf{u}}) \tag{77}$$

where

$$\dot{\Pi}[\bar{\mathbf{u}}, \dot{\mathbf{u}}] = \tfrac{1}{2}\dot{\mathbf{u}}^{\mathrm{T}}\mathbf{K}\dot{\mathbf{u}} - \dot{\mathbf{F}}_2^{\mathrm{T}}\dot{\mathbf{u}}_2 \tag{78}$$

is the incremental potential energy

$$\dot{\mathscr{K}}(u) = \{\dot{\mathbf{u}} \mid \nabla G_i(\mathbf{u}_c)\dot{\mathbf{u}}_c \leqslant 0\, \forall\, i \in B, \quad \nabla G_i(\bar{\mathbf{u}}_c)\dot{\mathbf{u}}_c = 0\, \forall\, i \in C\} \tag{79}$$

is the set of kinematically admissible incremental displacements and

$$\delta\dot{\mathscr{K}}(\bar{\mathbf{u}}, \dot{\mathbf{u}}) = \{\delta\dot{\mathbf{u}} \mid \nabla G_i(\bar{\mathbf{u}}_c)(\delta\dot{\mathbf{u}}_c + \dot{\mathbf{u}}_c) \leqslant 0\, \forall\, i \in B, \quad \nabla G_i(\bar{\mathbf{u}}_c)\delta\dot{\mathbf{u}}_c = 0\, \forall\, i \in C\} \tag{80}$$

is the set of kinematically admissible variations of displacement increments.

One can finally remark that Problem 5 may be interpreted as the principle of virtual work for increments and eqn (73) is the corresponding equilibrium equation.

## 6. STABILITY AND UNIQUENESS

Classically, an equilibrium state of a mechanical system is said to be stable if an arbitrary small disturbance results in a motion that is close to the equilibrium state. This dynamic definition of stability is difficult to apply directly in practice. However, it is easily shown that a sufficient condition for its validity in the case of conservative systems is given by the requirement of a strict local minimum of the potential energy. Also, for unconstrained systems with an arbitrary small damping, Koiter[21] showed that this energy criterion is also necessary for stability. Here general practice is followed (e.g. Ref. [22]) and the following axiom is adopted: a mechanical system is stable in an equilibrium configuration if and only if the potential energy has a strict local minimum. In this case of unilateral constraints the interpretation of the minimum is of course that only the kinematically admissible variations of the configuration need to give a larger value of the potential energy. That is, one has an inequality constrained minimum problem and therefore a non-classical problem of elastic stability. Fortunately, however, a second-order sufficient condition of constrained minimization[23] gives a characterization of the situation. This condition is used to establish the stability theorem.

*Stability theorem*: a displacement vector $\mathbf{u}$ and a contact force vector $\mathbf{P}$ that satisfies Problem 2 characterize a stable equilibrium state if

$$\mathbf{v}^{\mathrm{T}}\mathbf{H}[\mathbf{u}, \mathbf{P}]\mathbf{v} > 0 \tag{81}$$

for all non-zero $\mathbf{v}$ that belongs to

$$\bar{M} = \{\mathbf{v} \mid \nabla G_i(\mathbf{u}_c)\mathbf{v}_c = 0, \forall\, i \in \bar{C}, \quad \nabla G_i(\mathbf{u}_c)\mathbf{v}_c \leqslant 0, \forall\, i \in \bar{B}\} \tag{82}$$

where

$$\mathbf{H}[\mathbf{u},\mathbf{P}] = \nabla\mathbf{B}(\mathbf{u})^T\nabla^2 W(\mathbf{B}(\mathbf{u}))\nabla\mathbf{B}(\mathbf{u}) + \nabla W(\mathbf{B}(\mathbf{u}))^T\nabla^2\mathbf{B}(\mathbf{u}) - \left[\begin{array}{c|c} \mathbf{P}^T\nabla^2\mathbf{G}(\mathbf{u}_c) & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right] \tag{83}$$

$$\bar{C} = \{i \in I \,|\, G_i(\mathbf{u}_c) = 0, \quad P_i < 0\} \tag{84}$$

$$\bar{B} = \{i \in I \,|\, G_i(\mathbf{u}_c) = 0, \quad P_i = 0\}. \tag{85}$$

One can remark that if $\mathbf{u}$ and $\mathbf{P}$ coincide with $\bar{\mathbf{u}} = \mathbf{u}(t)$ and $\bar{\mathbf{P}} = \mathbf{P}(t)$ of Section 5, $\mathbf{H}[\mathbf{u},\mathbf{P}]$, $\bar{C}$ and $\bar{B}$ coincide with $\mathbf{K}$, $C$ and $B$, respectively. The significance of this will be explained shortly.

It is known that for unconstrained elastic structures the problem of stability is closely related to that of bifurcation, the nonuniqueness of the incremental response[24]. To investigate this connection in the case of unilateral constraints a sufficient condition for the uniqueness of Problem 5 is given. The variation of the incremental potential energy is written explicitly as

$$\delta\dot{\Pi}[\bar{\mathbf{u}},\dot{\mathbf{u}}] = \dot{\mathbf{u}}^T\mathbf{K}\delta\dot{\mathbf{u}} - \dot{\mathbf{F}}_2^T\delta\dot{\mathbf{u}}_2 \geqslant 0. \tag{86}$$

Assume that Problem 5 has two solutions $\mathbf{u}^1$ and $\mathbf{u}^2$. Let first $\dot{\mathbf{u}} = \dot{\mathbf{u}}^1$ and $\delta\dot{\mathbf{u}} = \dot{\mathbf{u}}^2 - \dot{\mathbf{u}}^1$, and secondly $\dot{\mathbf{u}} = \dot{\mathbf{u}}^2$ and $\delta\dot{\mathbf{u}} = \dot{\mathbf{u}}^1 - \dot{\mathbf{u}}^2$, in relation (86). By adding the two inequalities one obtains

$$(\dot{\mathbf{u}}^1 - \dot{\mathbf{u}}^2)^T\mathbf{K}(\dot{\mathbf{u}}^1 - \dot{\mathbf{u}}^2) \leqslant 0. \tag{87}$$

Clearly, this is a contradiction if $\mathbf{K}$ is positive definite on the subspace

$$M = \{\mathbf{v} \,|\, \nabla G_i(\bar{\mathbf{u}}_c)\mathbf{v}_c = 0 \quad \forall\, i \in C\}. \tag{88}$$

Thus, by comparing this result with the stability theorem it is found that even for unilaterally supported structures stability and bifurcation are related. However, the two sufficiency theorems do not coincide as in the case of unconstrained problems, where both $\bar{M}$ and $M$ equal the Euclidean space. The difference between the two theorems is reminiscent of the similar, now classical, situation of "stable bifurcation" encountered in the theory of elastic–plastic bodies[13]. The fact that the set of stable equilibrium configurations in the sense of the stability theorems does not coincide with the set of states such that the incremental response is unique, is reinforced by a simple counter-example in the next section.

Moreover, what seems to be of further significance in the present analysis is the presence of the term $\mathbf{v}_c^T\mathbf{P}^T\nabla^2\mathbf{G}(\mathbf{u}_c)\mathbf{v}_c$ in the quadratic form that determines stability and uniqueness. As noted in Section 5 this term is similar to the geometric stiffness matrix. Its presence generalizes the elementary example of stability of rigid bodies shown in Fig. 2(a) to the case of deformable bodies, Fig. 2(b). This feature of the theory will also be exemplified in the next section.

## 7. TWO EXAMPLES

In this last section two naturally discrete problems are discussed to illustrate two different aspects of the theory presented. In the first problem, the difference between the sufficiency theorems for stability and incremental uniqueness, respectively, is exemplified. In the second one, the significance of the curvature of the rigid surface is shown.

### 7.1. *An example showing bifurcation from a stable configuration*

Consider the two-dimensional system in Fig. 3, which is suggested by the model used by Shanley to investigate elastic–plastic column failure[12]. It consists of four linear springs with spring constants $k_1$ and $k_2$ and two rigid bars with lengths $2c$ and $l$. Point $D$ is restricted

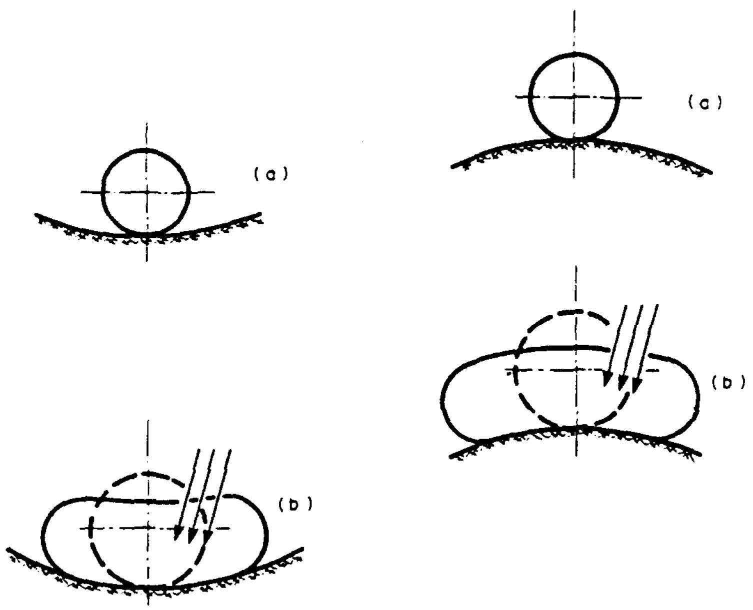


Fig. 2. (a) An elementary example of stable and unstable rigid body systems. (b) Generalization of the elementary example in the case of a deformable body.

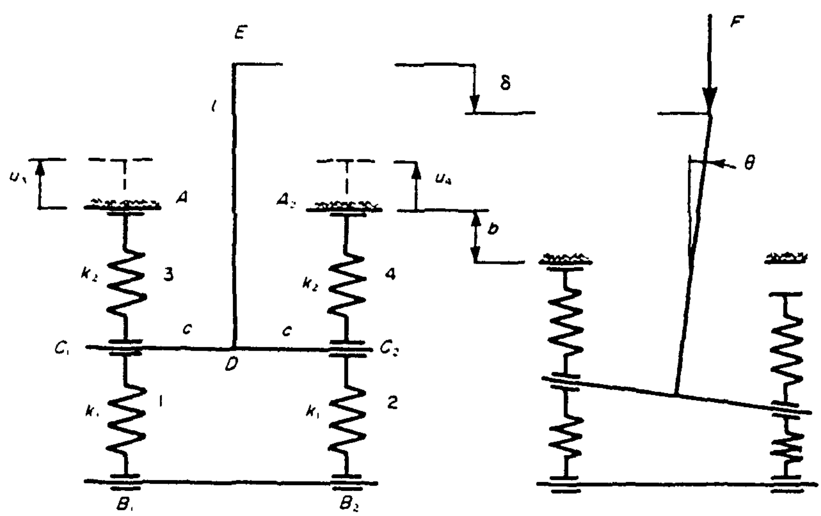


Fig. 3. Example showing bifurcation from a stable configuration.

to move vertically and at points $A$ one has unilateral supports. Thus, the problem has four degrees of freedom and as such the vertical displacement $\delta$ at point $E$, the rotation $\theta$ and the displacements $u_3$ and $u_4$ of the upper ends of springs 3 and 4 are chosen. The relation $\boldsymbol{\varepsilon} = \mathbf{B}(\mathbf{u})$, where $\boldsymbol{\varepsilon}$ contains the elongations of the springs and $\mathbf{u}$ contains the four parameters just mentioned, is easily obtained from geometric considerations. The relation $\mathbf{G}(\mathbf{u}_c) \leqslant \mathbf{0}$ is given by

$$\begin{bmatrix} b \\ b \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \end{bmatrix} \leqslant \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{89}$$

Thus, if $\delta = 0 = \theta$ springs 3 and 4 are compressed a distance $b$. Furthermore, the constitutive relation $\boldsymbol{\sigma} = \nabla W(\boldsymbol{\varepsilon})$ is linear and it is given by the matrix diag $[k_1, k_1, k_2, k_2]$.

For $\theta = 0$ it can be verified that the incremental equilibrium eqn (73) is

$$\{\mathbf{K}_1+\mathbf{G}\}\begin{bmatrix}\dot\delta\\ \dot\theta\\ \dot u_3\\ \dot u_4\end{bmatrix}=\begin{bmatrix}\dot F\\ 0\\ -\dot P_3\\ -\dot P_4\end{bmatrix} \tag{90}$$

where

$$\mathbf{K}_1=\begin{bmatrix}2(k_1+k_2) & 0 & k_2 & k_2\\ 0 & 2c^2(k_1+k_2) & -ck_2 & ck_2\\ k_2 & -ck_2 & k_2 & 0\\ k_2 & ck_2 & 0 & k_2\end{bmatrix} \tag{91}$$

and

$$\mathbf{G}=\text{diag}\,\{0, l[k_2(2\delta+u_3+u_4)-2k_1\delta], 0, 0\}. \tag{92}$$

To investigate the difference between bifurcation and stability the particular value $b = c^2/l$ is chosen for the initial negative gap at $A$. The reason for this is that the configuration $\delta = -u_3 = -u_4 = b$ and $\theta = 0$ is then a stable equilibrium configuration, which, nevertheless, does not fulfil the conditions for uniqueness of an incremental solution. In this configuration one has $\mathbf{G}(\mathbf{u}_c) = \mathbf{0}$ and $\mathbf{P} = \mathbf{0}$, so the set $M = R^4$ and the set

$$\bar M = \{(\dot\delta, \dot\theta, \dot u_3, \dot u_4)\,|\,\dot u_3 \leqslant 0, \dot u_4 \leqslant 0\}.$$

The stability theorem is verified by solving the homogeneous equilibrium problem in the configuration under investigation

$$\begin{bmatrix}2(k_1+k_2) & 0 & k_2 & k_2\\ 0 & 2c^2k_2 & -ck_2 & ck_2\\ k_2 & -ck_2 & k_2 & 0\\ k_2 & ck_2 & 0 & k_2\end{bmatrix}\begin{bmatrix}\dot\delta\\ \dot\theta\\ \dot u_3\\ \dot u_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \tag{93}$$

The solutions are

$$[\dot\delta, \dot\theta, \dot u_3, \dot u_4] = [0, \lambda, c\lambda, -c\lambda], \quad \lambda = \mathbb{R}. \tag{94}$$

Obviously, only for $\lambda = 0$ a solution that belongs to $\bar M$ is obtained and since the matrix in eqn (93) is positive semidefinite the stability theorem is satisfied.

However, the uniqueness condition is not satisfied and assuming $\dot P_3 = 0 = \dot P_4$ it is found that all solutions of eqn (90) can be written as

$$[\dot\delta, \dot\theta, \dot u_3, \dot u_4] = [0, \lambda, c\lambda, -c\lambda] + \frac{\dot F}{2k_1}[1, 0, 1, 1]. \tag{95}$$

The conditions $\dot u_3 \leqslant 0$ and $\dot u_4 \leqslant 0$ are satisfied for $|\lambda| \leqslant \dot F/2ck_2$ and for such values eqn (95) is a solution of the incremental problem. Thus, such a solution is not unique.

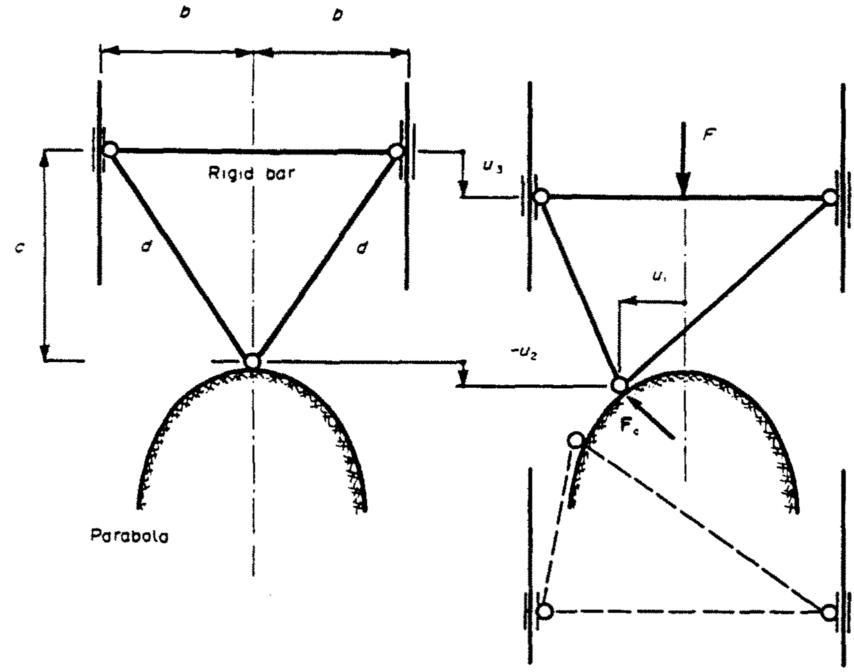


Fig. 4. (a) Example showing the influence of the curvature of the rigid support on instability and bifurcation. (b) Two post-bifurcation configurations.

### 7.2. *An example showing the influence of the curvature of the rigid support on instability and bifurcation*

Consider again a two-dimensional structure. It consists of two deformable and one rigid bar with initial lengths $d$ and $2b$, respectively, as shown in Fig. 4. Furthermore, there is a parabolic rigid surface present, i.e.

$$G(\mathbf{u}_c) = -(u_1^2 + ku_2) \tag{96}$$

where $k$ is a constant. For a material behaviour that is described by a linear relation

$$S = CE \tag{97}$$

between the Green's strain and the second Piola–Kirchhoff stress it can be shown (see p. 235 of Ref. [25]), that the following equilibrium equations hold for this problem:

$$F_{c1} = \frac{a_0 C}{d^3} u_1 [2(b^2 - c(u_2+u_3)) + u_1^2 + (u_2+u_3)^2] \tag{98}$$

$$F = F_{c2} = \frac{a_0 C}{d^3} (u_2 + u_3 - c) \, [(u_2+u_3)\,(u_2+u_3-2c) + u_1^2] \tag{99}$$

where $a_0$ is the initial cross-sectional area of a deformable bar. Note that eqn (99) represents two equations. Equations (98) and (99) can be put in non-dimensional form[26] by introducing

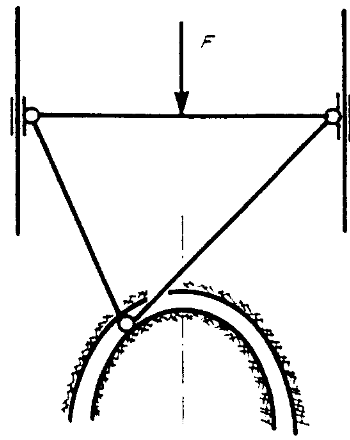


Fig. 5. A hypothetic bilaterally supported structure.

$$\bar{u}_1 = u_1/b, \quad \bar{u}_2 = u_2/b, \quad \bar{u}_3 = u_3/b$$

$$\bar{F}_{c1} = \frac{F_{c1}d^3}{a_0Cb^3}, \quad \bar{F}_{c2} = \frac{F_{c2}d^3}{a_0Cb^3}, \quad \bar{F} = \frac{Fd^3}{a_0Cb^3}.$$

One obtains

$$\bar{F}_{c1} = \bar{u}_1[2(1-\mu(\bar{u}_2+\bar{u}_3))+\bar{u}_1^2+(\bar{u}_2+\bar{u}_3)^2] \tag{100}$$

$$\bar{F} = \bar{F}_{c2} = (\bar{u}_2+\bar{u}_3-\mu)\,[(\bar{u}_2+\bar{u}_3)\,(\bar{u}_2+\bar{u}_3-2\mu)+\bar{u}_1^2] \tag{101}$$

where $\mu = c/b$. Likewise, eqn (96) can be put in non-dimensional form and, reinterpreting $k$ it can simply be considered as non-dimensional as it stands. In the sequel the bar notation will be dropped for non-dimensional variables in other equations as well.

The potential energy of the problem can then be written as

$$\Pi = \tfrac{1}{2}[u_1-\mu(u_2+u_3)+\tfrac{1}{2}(u_1^2+(u_2+u_3)^2)]^2 + \tfrac{1}{2}[-u_1+\mu(u_2+u_3)+\tfrac{1}{2}(u_1^2+(u_2+u_3)^2)]^2 - Fu_3. \tag{102}$$

Relation $\mathbf{F}_c = \nabla\mathbf{G}(\mathbf{u}_c)^T\mathbf{P}$ is given by

$$\begin{bmatrix} F_{c1} \\ F_{c2} \end{bmatrix} = -\begin{bmatrix} 2u_1 \\ k \end{bmatrix} P. \tag{103}$$

It now happens that for a hypothetic bilaterally supported structure, i.e. a structure according to Fig. 5 where $\mathbf{G}(\mathbf{u}_c) = \mathbf{0}$ and $\mathbf{P} \lesseqgtr \mathbf{0}$, eqns (100), (101) and (103) can be solved explicitly for all loads $\mathbf{F}$. One will proceed to do so.

First, consider the symmetrical case $u_1 = 0 = u_2$. Equation (100) is identically satisfied and eqn (101) gives

$$F = u_3(u_3-\mu)\,(u_3-2\mu). \tag{104}$$

That is, the solution is a cubic.

Secondly, for the asymmetrical case; $u_1 \neq 0$, which implies $u_2 \neq 0$, one obtains by introducing eqn (103) and the equality $F = F_{c2}$ in eqn (100)

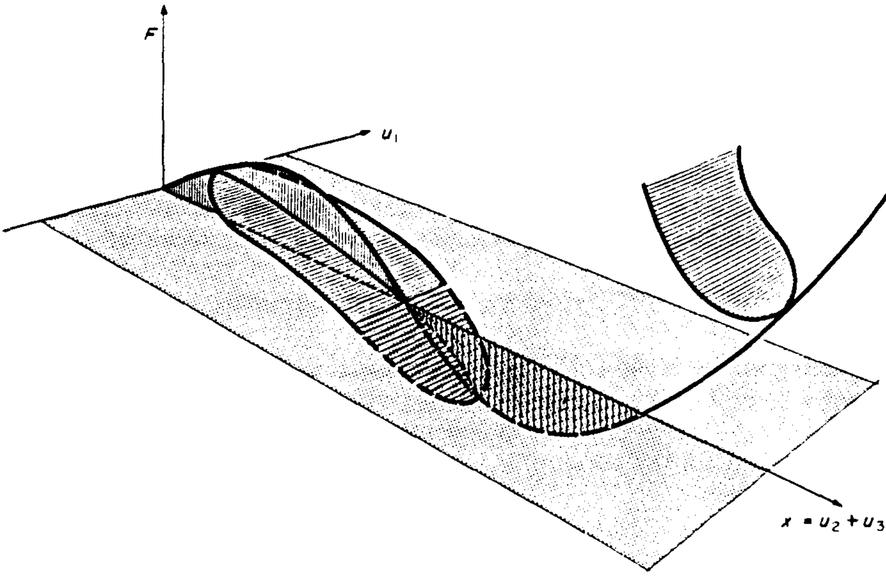


Fig. 6. Qualitative behaviour of equilibrium paths.

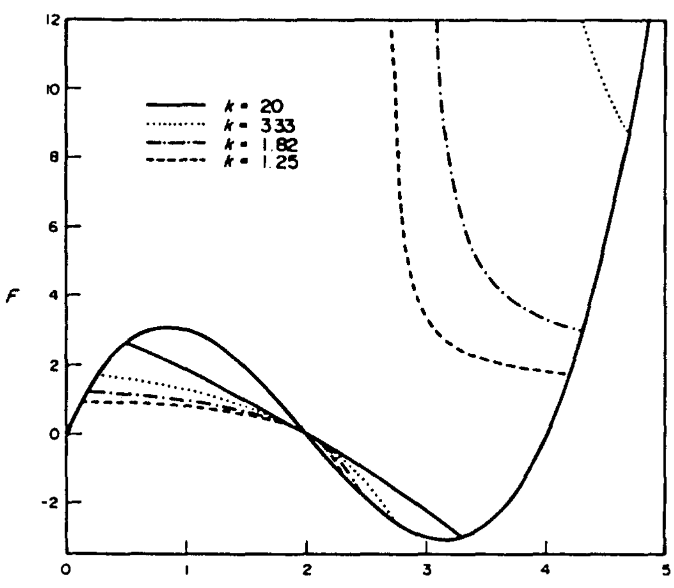


Fig. 7. The influence of $k$, i.e. the curvature of the rigid support.

$$u_1\left\{2\frac{F}{k}-[2(1-\mu x)+u_1^2+x^2]\right\}=0 \tag{105}$$

where $x = u_2+u_3$. By eliminating $u_1^2$ between eqns (101) and (105) one obtains

$$x=\frac{Fk}{2(F-k)}+\mu \tag{106}$$

which allows $u_1$ to be calculated from eqn (101) or eqn (105). One obtains

$$u_1^2 = \frac{8(F-k)^3 - F^2k^3 + \mu^2 4k(F-k)^2}{4k(F-k)^2} \tag{107}$$

Thus, when the right-hand side of this equation is non-negative there exists a post-bifurcation path (or rather, two paths) of solutions besides the primary path given by eqn (104).

The qualitative behaviour of the equilibrium paths is shown in Fig. 6, and in Fig. 7 the influence of $k$, i.e. the curvature of the rigid surface, is shown. The intersections between the three branches of the solution can be found by solving $F = F^*$ such that

$$8(F^*-k)^3 - F^{*2}k^3 + \mu^2 4k(F^*-k)^2 = 0 \tag{108}$$

that is, by letting $u_1^2 = 0$ in eqn (107). The part of the equilibrium paths corresponding to $F < 0$ is shown by broken lines in Fig. 6 since they exist only for the hypothetic structure.

If $k \to \infty$, the curvature of the rigid support approaches zero. In that case the first post-bifurcation path, for low values of $u_2 + u_3 = u_3$, can be shown to be given by[26]

$$u_1 = \pm(u_3(2\mu - u_3) - 2)^{1/2}. \tag{109}$$

On the other hand, the second post-bifurcation path, for high values of $u_2 + u_3$, disappears for this case. Thus, its existence is due to the contact force dependent term in the tangential stiffness matrix, as discussed at the end of Section 6.

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